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DYNAMIC STABILITY PROBLEM OF A NON-PRISMATIC ROD

P. Ruta

Institute of Civil Engineering, Wrocław University of Technology, Wrocław, Poland.
E-mail: ruta@i14odt.iil.pwr.wroc.pl

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A dynamic stiffness matrix for a non-prismatic rod finite element resting on a two-parameter non-homogenous elastic foundation has been determined. To obtain the solution the shape function was approximated by Chebyshev series. This yielded closed analytical formulae for the coefficients of the matrices sought. The finite element obtained was used to solve the dynamic stability problem for a non-prismatic cantilever column. The results were compared with those reported by other authors.

1. INTRODUCTION

The use of variable-cross-section rod systems in modern engineering structures has been increasing due to the necessity for the rational and economical design of structures and for architectural reasons. Solutions of many static problems, including stability problems, can be found in a monograph by Krynicki and Mazurkiewicz [1]. An analytical solution, consisting of the expansion of the displacement function into a Fourier series and the application of variational methods, was presented by Heidebrecht [2]. Fourier series, supplemented with power polynomials, were applied to solve linear, variable-coefficient differential equations (derived from, e.g., variable-cross-section beam vibration problems) in a paper by Ganga Rao and Spyrakos [3]. A rigidity and mass matrix for a beam with linearly variable height was determined by Gupta [4]. Non-prismatic beams were also studied by Eisenberger who in reference [5] determined stiffness matrix elements for several kinds of non-prismatic beams. Jointly with Reich [6], Eisenberger applied the finite element method to a static and dynamic analysis to solve the stability problem, approximating the beam's displacements by polynomials of degree 3. In reference [7], he presented formulae for stiffness matrix elements for a beam element with variable rigidities described by power series. Klasztorny [8] applied the same polynomial approximation to determine a stiffness and mass matrix for Euler and Timoshenko beam finite elements with variable parameters. Ruta [9] applied Chebyshev series to solve the vibration problem for a non-prismatic beam resting on a non-homogenous two-parameter elastic foundation. A solution in the form of a series relative to Chebyshev polynomials was obtained by solving an infinite system of algebraic equations for harmonic vibration. In the case of aperiodic vibration, an infinite system of ordinary differential equations had to be solved. The equations' coefficients were defined by closed analytic formulae.

Among works on stability, it is worth noting those dealing with systems under non-potential loads. Using Bessel functions Elishakoff and Pellegrni [10] solved the problem of a simply supported rod loaded with a tangential, distributed follower load.

Massey and Van der Meen [11] studied the stability of a non-prismatic cantilever under an applied follower force. Sankaren and Venkateswara Rao [12] determined the critical follower load values for taper columns with rigidly and elastically fixed ends. Glabisz [13] extended the Eisenberger formulae [7] for stiffness and mass matrices for rods with variable rigidity and density to rods resting on a two-parameter foundation and subjected to non-potential loads. He applied his method to solve several stability problems relating to non-prismatic cantilever rods subjected to distributed and concentrated non-potential loads. Similar to references [7, 8], power series were used to approximate displacement functions in reference [13].

In this paper, a non-prismatic finite rod element with variable strength and geometric parameters, resting on a two-parameter non-homogenous elastic foundation, is analyzed following the procedure in [14]. It is assumed that the rod's variable parameters, such as flexural rigidity, axial rigidity and density, the foundation's variable parameters and the load can be represented by an expansion into a series relative to Chebyshev polynomials of the first kind. Using the theorems and relationships for the above polynomials found in reference [15] and the results presented in reference [9], shape functions are determined and on their basis a dynamic rigidity matrix is derived for the analyzed element. This method is applied for one finite element to solve the stability problem for a non-prismatic cantilever rod and a clamped–simply supported rod, loaded with a concentrated or distributed follower load. The examples considered were taken from Glabisz"s paper [13]. The above element was also used to analyze a frame system subjected to axial potential loads. The obtained numerical results are compared with those reported in papers [1, 13, 17, 18].

2. PROBLEM FORMULATION

A non-prismatic, rectilinear Euler rod with a length of $2a$, resting on a two-parameter elastic foundation, subjected to normal $P(X,t)$ and tangential $R(X,t)$ loads (Figure 1) is considered. In addition, non-potential distributed load $s(X)$ and at the rod's ends, concentrated non-potential forces P_5, P_6 act axially on the rod.

In the case considered, the rod's transverse and longitudinal vibrations are described by the following partial differential equations:

$$\frac{\partial^2}{\partial X^2}\left(EJ(X)\frac{\partial^2 W}{\partial X^2}\right)-\frac{\partial}{\partial X}\left(N(X)\frac{\partial W}{\partial X}\right)-\frac{\partial}{\partial X}\left(C(X)\frac{\partial W}{\partial X}\right)+K(X)\,W(X)+\rho(x)\frac{\partial^2 W}{\partial t^2}$$
$$=P(X,t)+s(X)\,\eta(X)\frac{\partial W}{\partial X}, \tag{1}$$

$$-\frac{\partial}{\partial X}\left(EA(X)\frac{\partial U}{\partial X}\right)+F(X)\,U(X)+\rho(X)\frac{\partial^2 U}{\partial t^2}=R(X,t), \tag{2}$$

where W and U denote displacement, respectively, perpendicular and tangent to the rod's axis, E is Young's modulus, A and J are the rod's field and moment of inertia, ρ is mass per unit of length and $F(X)$, $K(X)$, $C(X)$ are functions characterizing the elastic foundation's reactions. A geometric interpretation of function coefficient $\eta(X)$, called a "follower coefficient", is shown in Figure 2. If $\eta = 0$, load $s(X)$ becomes a classic potential load with a spatially fixed direction and a materially prescribed point of application, whereas when $\eta = 1$, $s(X)$ is a strictly follower load.

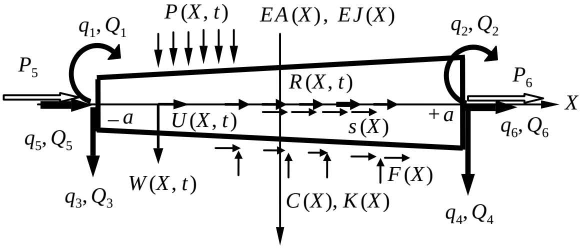

Figure 1. Diagram of non-prismatic rod.

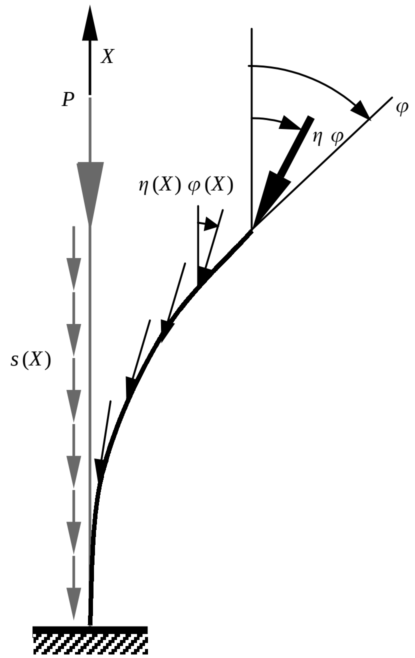

Figure 2. Geometric interpretation of follower coefficient η.

Cross-sectional forces: bending moments, the shearing forces and axial forces are defined as follows:

$$M(X,t) = -EJ\,\frac{\partial^2 W}{\partial X^2},$$

$$T(X,t) = -\frac{\partial}{\partial X}\left(EJ\,\frac{\partial^2 W}{\partial X^2}\right) + N\,\frac{\partial W}{\partial X},$$

$$Q(X,t) = EA\,\frac{\partial U}{\partial X}. \tag{3}$$

If discussions are limited to harmonic vibration and using relations

$$x = X/a,\; W(X,t) = W(X)\,\mathrm{e}^{\mathrm{i}\omega t} = a\,w(x)\,\mathrm{e}^{\mathrm{i}\omega t},\; U(X,t) = U(X)\,\mathrm{e}^{\mathrm{i}\omega t} = a\,u(x)\,\mathrm{e}^{\mathrm{i}\omega t},$$
$$P(X,t) = P(X)\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{P_0}{a}\;p(x)\,\mathrm{e}^{\mathrm{i}\omega t},\; R(X,t) = R(X)\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{P_0}{a}\;r(x)\,\mathrm{e}^{\mathrm{i}\omega t} \tag{4}$$

and

$$s(X)\eta(X) = \frac{P_0}{a}\,\bar{s}(x)\bar{\eta}(x) = \frac{P_0}{a}\,\bar{S}(x) \tag{5}$$

equations (1) and (2), give

$$\overline{EJ}(x)\frac{\partial^4 w}{\partial x^4} + \left(2\,\frac{\partial \overline{EJ}(x)}{\partial x}\right)\frac{\partial^3 w}{\partial x^3} + \left(\frac{\partial^2 \overline{EJ}(x)}{\partial x^2} - n(\overline{N}(x) + \overline{C}(x))\right)\frac{\partial^2 w}{\partial x^2} \tag{6}$$

$$- n\left(\frac{\partial \overline{N}(x)}{\partial x} + \frac{\partial \overline{C}(x)}{\partial x} + \overline{S}(x)\right)\frac{\partial w}{\partial x} + n\overline{K}(x)w - \omega^2 g\bar{\rho}(x)w = np(x),$$

$$- d\left(\overline{EA}(x)\frac{\partial^2 u}{\partial x^2} + \frac{\partial \overline{EA}(x)}{\partial x}\frac{\partial u}{\partial x}\right) + n\,\overline{F}(x)u - \omega^2 g\bar{\rho}(x)u = n\,r(x), \tag{7}$$

and cross-sectional forces (3) are expressed by the formulae

$$m(x) = \frac{M(ax)a}{EJ_0} = -\,\overline{EJ}\,\frac{\partial^2 w}{\partial x^2},$$

$$t(x) = \frac{T(ax)a^2}{EJ_0} = -\left(\frac{\partial}{\partial x}\,\overline{EJ}\right)\frac{\partial^2 w}{\partial x^2} - \overline{EJ}\frac{\partial^3 w}{\partial x^3} + n\overline{N}\frac{\partial w}{\partial x}, \tag{8}$$

$$q(x) = \frac{Q(ax)}{EA_0} = \overline{EA}\frac{\partial u}{\partial x},$$

where

$$EJ = EJ_0\overline{EJ}, \quad N = P_0\overline{N}, \quad C = P_0\overline{C}, \quad K = \frac{P_0}{a^2}\overline{K}, \quad \rho = \rho_0\,\bar{\rho},$$

$$EA = EA_0\,\overline{EA}, \quad F = \frac{P_0}{a^2}\overline{F}, \quad n = \frac{a^2 P_0}{EJ_0}, \quad g = \frac{a^4\rho_0}{EJ_0}, \quad d = \frac{a^2\,EA_0}{EJ_0}, \tag{9}$$

and EJ_0, EA_0, ρ_0, P_0 are reference values.

To simplify the notation, assume EJ, EA, N, S, C, K, ρ, F for $\overline{EJ}$, $\overline{EA}$, $\overline{N}$, $\overline{S}$, $\overline{C}$, $\overline{K}$, $\bar{\rho}$, $\overline{F}$.

3. SOLUTION OF THE PROBLEM

Solutions of the differential equations (6) and (7) are sought, having the form of a Chebyshev series of the first kind

$$w(x) = \sum_{l=0}^{\infty}{}'\, a_l[w]\; T_l(x) = \sum_{l=0}^{\infty}{}'\, w_l\; T_l(x), \tag{10}$$

$$u(x) = \sum_{l=0}^{\infty}{}'\, a_l[u]\; T_l(x) = \sum_{l=0}^{\infty}{}'\, u_l\; T_l(x), \tag{11}$$

where

$$\sum_{l=0}^{\infty}{}'\; a_l[f] = \tfrac{1}{2}\,a_0[f] + a_1[f] + a_2[f] + \cdots, \tag{12}$$

and $a_l[w]$, $a_l[u]$ are unknown coefficients of the expansion of the displacement functions w and u into a Chebyshev series, denoted further on in the paper as w_l and u_l, respectively.

To solve equations (6) and (7) the following theorem on ordinary differential equations [15] will be applied:

Theorem. *If a function f satisfies a differential linear equation of order $n > 0$*

$$\sum_{m=0}^{n} \hat{P}_m(x)\, f^{(n-m)}(x) = \hat{P}(x), \tag{13}$$

and

$$Q_m(x) = \sum_{j=0}^{m} (-1)^{m+j} \binom{n-j}{m-j} \hat{P}_j^{(m-j)}(x),\ m = 0, 1, \ldots, n, \tag{14}$$

and the Chebyshev series coefficients in functions $(Q_0 f)^{(n)}, (Q_1 f)^{(n-1)}, \ldots, Q_n f, \hat{P}$ are determined, then for each integer k the following identity is true:

$$\sum_{m=0}^{n} 2^{n-m} \sum_{j=0}^{m} b_{nmj}(k)\, a_{k-m+2j}[Q_m(x) f(x)] = \sum_{j=0}^{n} b_{nnj}(k)\, a_{k-n+2j}[\hat{P}(x)], \tag{15}$$

where $b_{nmj}(k)$ are polynomials of integer variable k.

$$b_{nmj}(k) = (-1)^j \binom{m}{j} (k-n)_{n-m+j} (k-m+2j)(k+j+1)_{n-j} (k^2-n^2)^{-1},$$

$$m = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m. \tag{16}$$

$$(c)_k = \begin{cases} 1 & \text{for } k = 0, \\ c(c+1)(c+2)\ldots.(c+k-1) & \text{for } k = 1, 2, \end{cases} \tag{17}$$

and $a_k[h]$ is the kth coefficient of the expansion of the function $h(x)$ into a Chebyshev series relative to Chebyshev polynomials of the first kind (a proof of this theorem can be found in reference [15], pp. 231–234).

Functions $\hat{P}_m, \hat{P}$ in equations (6) and (7) for displacements w and u are given by the formulae

$$\hat{P}_0(x) = EJ(x), \qquad \hat{P}_1(x) = 2\frac{\partial EJ(x)}{\partial x}, \qquad \hat{P}_2(x) = \frac{\partial^2 EJ(x)}{\partial x^2} - n(N(x) + C(x)),$$

$$P_3(x) = -n\left(\frac{\partial N(x)}{\partial x} + \frac{\partial C(x)}{\partial x} + S(x)\right), \quad \hat{P}_4(x) = n\,K(x) - \omega^2 g\,\rho(x), \qquad \hat{P}(x) = np(x) \tag{18}$$

and

$$\hat{P}_0(x) = -d\,EA(x), \qquad \hat{P}_1(x) = -d\frac{\partial EA(x)}{\partial x},$$
$$\hat{P}_2(x) = nF(x) - \omega^2 g\,\rho(x), \qquad \hat{P}(x) = nr(x). \tag{19}$$

After relation (14) is applied, functions Q_m associated with $\hat{P}_m$ assume, for w and u, respectively, the following form:

$$\begin{aligned} Q_0(x) &= EJ(x), \\ Q_1(x) &= -2\frac{\partial EJ(x)}{\partial x}, \\ Q_2(x) &= \frac{\partial^2 EJ(x)}{\partial x^2} - n(N(x) + C(x)), \\ Q_3(x) &= n\left(\frac{\partial N(x)}{\partial x} + \frac{\partial C(x)}{\partial x} + S(x)\right), \\ Q_4(x) &= n\left(K(x) - \frac{\partial S(x)}{\partial x}\right) - \omega^2 g\,\rho(x), \end{aligned} \tag{20}$$

and

$$\begin{aligned} Q_0(x) &= -\,d\,EA(x), \\ Q_1(x) &= d\,\frac{\partial EA(x)}{\partial x}, \\ Q_2(x) &= n\,F(x) - \omega^2 g\,\rho(x). \end{aligned} \tag{21}$$

Using the above theorem, formulae (20) and (21) and the following relations (see reference [15], p. 128, equation (33), p. 124, equation (17)):

$$a_k[f(x)\cdot g(x)] = \frac{1}{2}\sum_{l=0}^{\infty}{}' \, a_l[f]\,(a_{k-l}[g] + a_{k+l}[g]), \tag{22}$$

$$a_l = \frac{1}{2l}(a'_{l-1} - a'_{l+1}), \quad l \neq 0, \tag{23}$$

where $a_l = a_l[f]$ and $a'_l = a_l[f']$, after transformations infinite systems of algebraic equations are obtained which allow one to determine coefficients w_l of the expansion of displacement function w

$$\begin{aligned} \sum_{l=0}^{\infty}{}' \, \{ & 8(k^2-9)(k^2-4)\,l\,[(k+1)(l-1)\,e_{k-l} - 2\sum_{j=1}^{l-1}(k-l+2j)\,e_{k-l+2j} \\ & + (k-1)(l-1)\,e_{k+l}] \\ & - 2\,n(k^2-9)\,l\,[(k+1)(k+2)\,(n_{k-l-2} - n_{k+l-2}) - 2(k^2-4)(n_{k-l} - n_{k+l}) \\ & + (k-1)(k-2)\,(n_{k-l+2} - n_{k+l+2})] \\ & - 2n(k^2-9)\,l\,[(k+1)(k+2)\,(c_{k-l-2} - c_{k+l-2}) - 2(k^2-4)\,(c_{k-l} - c_{k+l}) \\ & + (k-1)(k-2)\,(c_{k-l+2} - c_{k+l+2})] \\ & + \tfrac{1}{2}n[(k+1)(k+2)(k+3)(k_{k-l-4} + k_{k+l-4}) - 4(k+3)(k^2-4)\,(k_{k-l-2} + k_{k+l-2}) \\ & + 6k(k^2-9)\,(k_{k-l} + k_{k+l}) - 4(k-3)(k^2-4)\,(k_{k-l+2} + k_{k+l+2}) \\ & + (k-1)(k-2)(k-3)\,(k_{k-l+4} + k_{k+l+4})] \end{aligned}$$

$$- nl\, [(k+1)(k+2)(k+3)(s_{k-l-3} - s_{k+l-3}) - 3(k+2)(k^2-9)\,(s_{k-l-1} - s_{k+l-1})$$

$$+ 3(k-2)(k^2-9)\,(s_{k-l+1} - s_{k+l+1}) - (k-1)(k-2)(k-3)(s_{k-l+3} - s_{k+l+3})]$$

$$- \tfrac{1}{2}\omega^2 g[(k+1)(k+2)(k+3)(g_{k-l-4} + g_{k+l-4}) - 4(k+3)(k^2-4)(g_{k-l-2} + g_{k+l-2})$$

$$+ 6\mathrm{k}(k^2-9)(g_{k-l} + g_{k+l}) - 4(k-3)(k^2-4)(g_{k-l+2} + g_{k+l+2}) + (k-1)(k-2)$$

$$\times (k-3)(g_{k-l+4} + g_{k+l+4})\}\; w_l$$

$$= n\{(k+1)(k+2)(k+3)p_{k-4} - 4(k+3)(k^2-4)p_{k-2} + 6k(k^2-9)p_k$$

$$- 4(k-3)(k^2-4)p_{k+2} + (k-1)(k-2)(k-3)p_{k+4}\}, \quad k = 0, 1, 2, 3, \ldots \tag{24}$$

and coefficients u_l of the expansion of displacement function u

$$\sum_{l=0}^{\infty}{}' \; \{ - 2d\,(k^2-1)\, l\,(d_{k-l} + d_{k+l})$$

$$+ \tfrac{1}{2} n[(k+1)(f_{k-l-2} + f_{k+l-2}) - 2k(f_{k-l} + f_{k+l}) + (k-1)(f_{k-l+2} + f_{k+l+2})]$$

$$- \tfrac{1}{2}\omega^2 g[(k+1)(g_{k-l-2} + g_{k+l-2}) - 2k(g_{k-l} + g_{k+l}) + (k-1)(g_{k-l+2} + g_{k+l+2})]\}u_l$$

$$= (k+1)\, r_{k-2} - 2k\, r_k + (k-1)\, r_{k+2}, \quad k = 0, 1, 2, 3, \ldots . \tag{25}$$

Parameters e_l, n_l, c_l, k_l, s_l, g_l, p_l and d_l, f_l, r_l in formulae (24) and (25) are coefficients of the expansions of the following functions which occur in equations (6) and (7):

$$EJ(x) = \sum_{l=0}^{\infty}{}' \; e_l T_l(x), \qquad N(x) = \sum_{l=0}^{\infty}{}' \; n_l T_l(x),$$

$$C(x) = \sum_{l=0}^{\infty}{}' \; c_l T_l(x), \qquad K(x) = \sum_{l=0}^{\infty}{}' \; k_l T_l(x), \tag{26}$$

$$S(x) = \sum_{l=0}^{\infty}{}' \; s_l T_l(x), \qquad \rho(x) = \sum_{l=0}^{\infty}{}' \; g_l T_l(x),$$

$$p(x) = \sum_{l=0}^{\infty}{}' \; p_l T_l(x),$$

$$EA(x) = \sum_{l=0}^{\infty}{}' \; d_l T_l(x), \quad F(x) = \sum_{l=0}^{\infty}{}' \; f_l T_l(x), \quad r(x) = \sum_{l=0}^{\infty}{}' \; r_l T_l(x). \tag{27}$$

The full derivation of equations (24) and (25) can be found in reference [9], but the equivalents of formulae (24) and (25) there contain editorial errors and the part of formula (24) relating to function $C(x)$ was derived incorrectly.

To calculate the displacements, angles of rotation and internal forces (see formula (8)) at the rod's ends, Chebyshev expansions of the functions $EJ(x)$, $N(x)$, $EA(x)$ (formulae (26) and

(27)) and the following relations ([15] p. 48, equations (14), (16)) will be used:

$$\begin{aligned} &T_n(1) = 1, && T_n(-1) = (-1)^n, \\ &T'_n(1) = n^2, && T'_n(-1) = -(-1)^n n^2, \\ &T''_n(1) = n^2(n^2-1)/3, && T''_n(-1) = (-1)^n n^2(n^2-1)/3, \\ &T'''_n(1) = n^2(n^2-1)(n^2-4)/15, && T'''_n(-1) = -(-1)^n n^2(n^2-1)(n^2-4)/15. \end{aligned} \tag{28}$$

After putting the values of polynomials $T_n(x)$ and their derivatives at points ± 1 into the formulae for the expansion of function EJ, N, EA, $\partial EJ/\partial x$, the following is obtained:

$$\begin{aligned} &EJ(+1) = EJ_+ = \sum_{l=0}^{\infty}{}' e_l T_l(1) = \sum_{l=0}^{\infty}{}' e_l, \\ &EJ(-1) = EJ_- = \sum_{l=0}^{\infty}{}' e_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l e_l, \\ &N(+1) = N_+ = \sum_{l=0}^{\infty}{}' n_l T_l(1) = \sum_{l=0}^{\infty}{}' n_l, \\ &N(-1) = N_- = \sum_{l=0}^{\infty}{}' n_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l n_l, \\ &\left.\frac{\partial EJ}{\partial x}\right|_{x=+1} = EJ'_+ = \sum_{l=0}^{\infty}{}' e_l T'_l(1) = \sum_{l=0}^{\infty}{}' l^2 e_l, \\ &\left.\frac{\partial EJ}{\partial x}\right|_{x=-1} = EJ'_- = \sum_{l=0}^{\infty}{}' e_l T'_l(-1) = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 e_l, \\ &EA(+1) = EA_+ = \sum_{l=0}^{\infty}{}' d_l T_l(1) = \sum_{l=0}^{\infty}{}' d_l, \\ &EA(-1) = EA_- = \sum_{l=0}^{\infty}{}' d_l T_l(-1) = \sum_{l=0}^{\infty}{}' (-1)^l d_l. \end{aligned} \tag{29}$$

If formulae (8) and the calculated values of polynomials $T_n(\pm 1)$ and functions (29) are used, relations for determining the displacement, angles of rotation and cross-sectional forces at the rod's end are obtained. These relations for transverse vibration and longitudinal vibration, respectively, are as follows:

$$\begin{aligned} &w(+1) = w_+ = \sum_{l=0}^{\infty}{}' w_l, w(-1) = w_- = \sum_{l=0}^{\infty}{}' (-1)^l w_l, \\ &\phi(+1) = \phi_+ = \sum_{l=0}^{\infty}{}' l^2 w_l, \ \phi(-1) = \phi_- = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 w_l, \\ &m(+1) = m_+ = -EJ_+ \frac{1}{3} \sum_{l=0}^{\infty}{}' l^2(l^2-1) w_l, \\ &m(-1) = m_- = -EJ_- \frac{1}{3} \sum_{l=0}^{\infty}{}' (-1)^l l^2(l^2-1) w_l, \\ &t(+1) = t_+ = -\sum_{l=0}^{\infty}{}' l^2 \left[\frac{1}{3}(l^2-1)EJ'_+ + \frac{1}{15}(l^2-1)(l^2-4)EJ_+ - nN_+ \right] w_l, \\ &t(-1) = t_- = \sum_{l=0}^{\infty}{}' (-1)^l l^2 \left[-\frac{1}{3}(l^2-1)EJ'_- + \frac{1}{15}(l^2-1)(l^2-4)EJ_- - nN_- \right] w_l \end{aligned} \tag{30}$$

and

$$u(+1) = u_+ = \sum_{l=0}^{\infty}{}' \; u_l, \quad u(-1) = u_- = \sum_{l=0}^{\infty}{}' \; (-1)^l u_l,$$
$$q(+1) = q_+ = EA_+ \sum_{l=0}^{\infty}{}' \; l^2 u_l, \quad q(-1) = q_- = -EA_- \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 u_l. \tag{31}$$

In infinite systems of algebraic equations (24) and (25), depending on the order of differential equation n to which they relate to, the first n equations for $k = 0, 1,..., n-1$ is satisfied identity-wise. The latter equations are replaced with equations for boundary conditions.

The infinite systems of equations can be presented in the following matrix form:

$$\left(\begin{bmatrix} \mathbf{A}_{pp} & \mathbf{A}_{pr} \\ \mathbf{A}_{rp} & \mathbf{A}_{rr} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{rp} & \mathbf{B}_{rr} \end{bmatrix} \right) \begin{bmatrix} \mathbf{w}_p \\ \mathbf{w}_r \end{bmatrix} = \begin{bmatrix} \mathbf{C}_p \\ \mathbf{C}_r \end{bmatrix}, \tag{32}$$

where submatrices $\mathbf{A}_{pp}$, $\mathbf{A}_{pr}$ have dimensions, respectively, $n \times n$ and $n \times \infty$ ($n = 4$ or 2) and their elements are coefficients relating to boundary conditions; submatrices $\mathbf{A}_{rp}$, $\mathbf{A}_{rr}$ and $\mathbf{B}_{rp}$, $\mathbf{B}_{rr}$ are matrices of the coefficients which occur in equation (24) or (25); $\mathbf{w}_p = [w_0,...,w_{n-1}]^{\mathrm{T}}$, $\mathbf{w}_r = [w_n, w_{n+1}, w_{n+2},...]^{\mathrm{T}}$; and vectors $\mathbf{C}_p$, $\mathbf{C}_r$ define boundary conditions and coefficients associated with an external load.

The elements of a rigidity matrix for a finite element

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}^w & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^u \end{bmatrix}, \tag{33}$$

are functions of variable ω and they can be determined from the following relation [16]:

$$\begin{aligned} \mathbf{S}^w = {} & \frac{1}{a^3} \int_{-1}^{1} \mathbf{H}''_w(x) \cdot EJ(x) \cdot \mathbf{H}''^{\mathrm{T}}_w(x) \mathrm{d}x + \frac{1}{a} \int_{-1}^{1} \mathbf{H}'_w(x) \cdot N(x) \cdot \mathbf{H}'^{\mathrm{T}}_w(x) \, \mathrm{d}x \\ & + a \int_{-1}^{1} \mathbf{H}_w(x) \cdot K(x) \cdot \mathbf{H}^{\mathrm{T}}_w(x) \mathrm{d}x + \frac{1}{a} \int_{-1}^{1} \mathbf{H}'_w(x) \cdot C(x) \cdot \mathbf{H}'^{\mathrm{T}}_w(x) \, \mathrm{d}x \quad , \\ & - a\omega^2 \int_{-1}^{1} \mathbf{H}_w(x) \cdot \rho(x) \cdot \mathbf{H}^{\mathrm{T}}_w(x) \mathrm{d}x - \int_{-1}^{1} \mathbf{H}'_w(x) \cdot S(x) \cdot \mathbf{H}^{\mathrm{T}}_w(x) \, \mathrm{d}x \end{aligned} \tag{34}$$

$$\begin{aligned} \mathbf{S}^u = {} & \frac{1}{a} \int_{-1}^{1} \mathbf{H}'_u(x) \cdot EA(x) \cdot \mathbf{H}'^{\mathrm{T}}_u(x) \mathrm{d}x + a \int_{-1}^{1} \mathbf{H}_u(x) \cdot F(x) \cdot \mathbf{H}^{\mathrm{T}}_u(x) \, \mathrm{d}x \\ & - a\omega^2 \int_{-1}^{1} \mathbf{H}_u(x) \cdot \rho(x) \cdot \mathbf{H}^{\mathrm{T}}_u(x) \, \mathrm{d}x, \end{aligned} \tag{35}$$

where matrices $\mathbf{H}_w(x)$, $\mathbf{H}_u(x)$ define relationships between displacements $W(x)$ and $U(x)$ in the element and co-ordinates $q_1, q_2, \ldots, q_6$ describing the displacements and rotations at the rod ends. The relationships are as follows:

$$W(x) = \mathbf{H}_w^{\mathrm{T}}(x) \cdot \mathbf{q}_w = [H_1(x)\ H_2(x)\ H_3(x)\ H_4(x)] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}, \tag{36}$$

$$U(x) = \mathbf{H}_u^{\mathrm{T}}(x) \cdot \mathbf{q}_u = [H_5(x) \quad H_6(x)] \begin{bmatrix} q_5 \\ q_6 \end{bmatrix}. \tag{37}$$

Functions $H_i(x)$, $i = 1,2,\ldots,6$ in formulae (36) and (37) are called shape functions. Functions H_1, H_2, H_3, H_4 can be determined by solving an infinite system of equations (24) in which the first four equations are as follows:

$$\begin{aligned} &\phi(-1)) = -\sum_{l=0}^{\infty}{}' (-1)^l l^2 w_l = q_1, \quad \phi(+1) = \sum_{l=0}^{\infty}{}' l^2 w_l = q_2, \\ &w(-1) = \sum_{l=0}^{\infty}{}' (-1)^l w_l = q_3/a, \quad w(+1) = \sum_{l=0}^{\infty}{}' w_l = q_4/a, \end{aligned} \tag{38}$$

which fulfil the following boundary conditions:

$$\begin{aligned} &-\text{for } H_1: \quad q_1 = \phi(-1) = 1, \quad q_2 = \phi(1) = 0, \quad q_3 = aw(-1) = 0, \quad q_4 = aw(1) = 0, \\ &-\text{for } H_2: \quad q_1 = \phi(-1) = 0, \quad q_2 = \phi(1) = 1, \quad q_3 = aw(-1) = 0, \quad q_4 = aw(1) = 0, \\ &-\text{for } H_3: \quad q_1 = \phi(-1) = 0, \quad q_2 = \phi(1) = 0, \quad q_3 = aw(-1) = 1, \quad q_4 = aw(1) = 0, \\ &-\text{for } H_4: \quad q_1 = \phi(-1) = 0, \quad q_2 = \phi(1) = 0, \quad q_3 = aw(-1) = 0, \quad q_4 = aw(1) = 1. \end{aligned} \tag{39}$$

As a result, four coefficient series $\mathsf{W}_l^k, k = 1,2,3,4;\ l = 0,1,2,\ldots$ defining the sought exact shape functions are obtained:

$$\begin{aligned} H_k(x) &= a\sum_{l=0}^{\infty}{}' \mathsf{W}_l^k T_l(x), \quad k = 1,2; \\ H_k(x) &= \hat{a}\sum_{l=0}^{\infty}{}' \mathsf{W}_l^k T_l(x), \quad k = 3,4; \end{aligned} \tag{40}$$

where $\hat{a}$ denotes a non-dimensional part of reference value a (e.g., if $a = 2{\cdot}5$ m, then $\hat{a} = 2{\cdot}5$).

Similarly one can determine H_5, H_6. In this case, the system of equations (25) are solved in which the first two equations are as follows:

$$u(-1) = \sum_{l=0}^{\infty}{}' (-1)^l u_l = q_5/a, \quad u(+1) = \sum_{l=0}^{\infty}{}' u_l = q_6/a, \tag{41}$$

and the boundary conditions are given by the formula

$$\begin{aligned} &\text{for } H_5: \quad q_5 = au(-1) = 1, \quad q_6 = au(1) = 0, \\ &\text{for } H_6: \quad q_5 = au(-1) = 0, \quad q_6 = au(1) = 1. \end{aligned} \tag{42}$$

If there are two solutions of system (25): U_l^k, $k = 5, 6$; $l = 0, 1, 2, \ldots$ then

$$H_k(x) = \hat{a} \sum_{l=0}^{\infty}{}' \; \mathsf{U}_l^k T_l(x), \quad k = 5, 6. \tag{43}$$

A dynamic rigidity matrix for a finite element can also be determined by directly calculating the values of forces $Q_1, Q_2, \ldots, Q_6$. The forces can be determined from formulae $(30)_{5\text{–}8}$

$$Q_1 = \frac{EJ_0}{a} m(-1) = -\frac{1}{3}\frac{EJ_0}{a} EJ_- \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 (l^2 - 1) w_l,$$

$$Q_2 = -\frac{EJ_0}{a} m(+1) = \frac{1}{3}\frac{EJ_0}{a} EJ_+ \sum_{l=0}^{\infty}{}' \; l^2 (l^2 - 1) w_l,$$

$$Q_3 = -\frac{EJ_0}{a^2} t(-1) = -\frac{EJ_0}{a^2} \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 \left[-\frac{1}{3}(l^2 - 1) EJ'_- + \frac{1}{15}(l^2 - 1)(l^2 - 4) EJ_- - nN_- \right] w_l$$

$$Q_4 = \frac{EJ_0}{a^2} t(+1) = -\frac{EJ_0}{a^2} \sum_{l=0}^{\infty}{}' \; l^2 \left[\frac{1}{3}(l^2 - 1) EJ'_+ + \frac{1}{15}(l^2 - 1)(l^2 - 4) EJ_+ - nN_+ \right] w_l \tag{44}$$

and $(31)_{3\text{–}4}$

$$Q_5 = -EA_0 q(-1) = EA_0 EA_- \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 u_l,$$
$$Q_6 = EA_0 q(+1) = EA_0 EA_+ \sum_{l=0}^{\infty}{}' \; l^2 u_l. \tag{45}$$

If shape function expansion coefficients H_1, H_2, H_3, H_4 and H_5, H_6, i.e., coefficients W_l^k, $k = 1, 2, 3, 4$; $l = 0, 1, 2, \ldots$ and U_l^k, k = 5, 6; $l = 0, 1, 2, \ldots$, are put in formulae (44) and (45), the following dynamic rigidity matrix elements are obtained:

$$S_{1k}^w = -\frac{1}{3}\frac{EJ_0}{a} EJ_- \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 (l^2 - 1) \mathsf{W}_l^k,$$

$$S_{2k}^w = \frac{1}{3}\frac{EJ_0}{a} EJ_+ \sum_{l=0}^{\infty}{}' \; l^2 (l^2 - 1) \mathsf{W}_l^k,$$

$$S_{3k}^w = -\frac{EJ_0}{a^2} \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 \left[-\frac{1}{3}(l^2 - 1) EJ'_- + \frac{1}{15}(l^2 - 1)(l^2 - 4) EJ_- - nN_- \right] \mathsf{W}_l^k, \tag{46}$$

$$S_{4k}^w = -\frac{EJ_0}{a^2} \sum_{l=0}^{\infty}{}' \; l^2 \left[\frac{1}{3}(l^2 - 1) EJ'_+ + \frac{1}{15}(l^2 - 1)(l^2 - 4) EJ_+ - nN_+ \right] \mathsf{W}_l^k,$$

$k = 1, 2, 3, 4,$

$$S_{5k}^u = EA_0 \, EA_- \sum_{l=0}^{\infty}{}' \; (-1)^l l^2 \mathsf{U}_l^k,$$
$$S_{6k}^u = EA_0 \, EA_+ \sum_{l=0}^{\infty}{}' \; l^2 \mathsf{U}_l^k, \; k = 5, 6. \tag{47}$$

If there is a non-potential, axial, concentrated load P_5, P_6 acting on the rod end, one should modify elements S^w_{31}, S^w_{42} which then assume the following form:

$$S^w_{31} = -\frac{EJ_0}{a^2}\sum_{l=0}^{\infty}{}' (-1)^l l^2 \left[-\frac{1}{3}(l^2-1)EJ'_- + \frac{1}{15}(l^2-1)(l^2-4)EJ_- - nN_- \right] \mathsf{W}^k_l - \eta P_5,$$

$$S^w_{42} = -\frac{EJ_0}{a^2}\sum_{l=0}^{\infty}{}' l^2 \left[\frac{1}{3}(l^2-1)EJ'_+ + \frac{1}{15}(l^2-1)(l^2-4)EJ_+ - nN_+ \right] \mathsf{W}^k_l - \eta P_6. \tag{48}$$

Obviously, forces P_5 , P_6 also influence the values of axial forces N_-, N_+.

The vectors of nodal active forces can be determined from the relation

$$\mathbf{F}^w = P_0 \int_{-1}^{1} p(x)\,\mathbf{H}_w(x)\,\mathrm{d}x, \quad \mathbf{F}^u = P_0 \int_{-1}^{1} r(x)\,\mathbf{H}_u(x)\,\mathrm{d}x. \tag{49}$$

After the expansions of functions $p(x)$, $r(x)$ and the shape function expansions are put in formula (49) and relation (22) is applied, this gives

$$F^w_i = -aP_0 \sum_{n=0}^{\infty}{}' \int_{-1}^{1} \left[\sum_{l=0}^{\infty}{}' p_l(\mathsf{W}^i_{n-l} + \mathsf{W}^i_{n+l}) \right] T_n(x),\ i = 1, 2;$$

$$F^w_i = -\hat{a}P_0 \sum_{n=0}^{\infty}{}' \int_{-1}^{1} \left[\sum_{l=0}^{\infty}{}' p_l(\mathsf{W}^i_{n-l} + \mathsf{W}^i_{n+l}) \right] T_n(x),\ i = 3, 4; \tag{50}$$

$$F^u_i = -\hat{a}P_0 \sum_{n=0}^{\infty}{}' \int_{-1}^{1} \left[\sum_{l=0}^{\infty}{}' r_l(\mathsf{U}^i_{n-l} + \mathsf{U}^i_{n+l}) \right] T_n(x),\ i = 5, 6.$$

If the following relation ([15], p. 43 equation (102))

$$\int_{-1}^{1} T_n(x)\,\mathrm{d}x = \begin{cases} -2/(n^2-1) & \text{for even } n, \\ 0 & \text{for odd } n \end{cases} \tag{51}$$

is applied, it is ultimately gives

$$F^w_i = -aP_0 \sum_{n=0}^{\infty}{}' \left[\sum_{l=0}^{\infty}{}' p_l(\mathsf{W}^i_{2n-l} + \mathsf{W}^i_{2n+l}) \right] \frac{1}{4n^2-1}, \quad i = 1, 2;$$

$$F^w_i = -\hat{a}P_0 \sum_{n=0}^{\infty}{}' \left[\sum_{l=0}^{\infty}{}' p_l(\mathsf{W}^i_{2n-l} + \mathsf{W}^i_{2n+l}) \right] \frac{1}{4n^2-1}, \quad i = 3, 4; \tag{52}$$

$$F^u_i = -\hat{a}P_0 \sum_{n=0}^{\infty}{}' \left[\sum_{l=0}^{\infty}{}' r_l(\mathsf{U}^i_{2n-l} + \mathsf{U}^i_{2n+l}) \right] \frac{1}{4n^2-1}, \quad i = 5, 6.$$

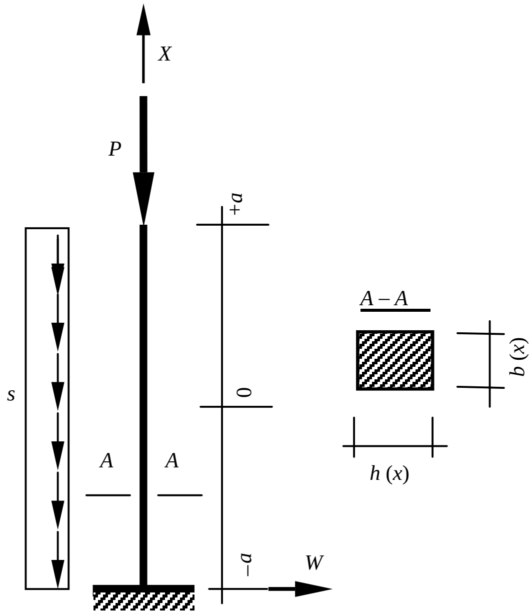

Figure 3. Cantilever rod loaded with non-potential concentrated force P and uniformly distributed non-potential load s.

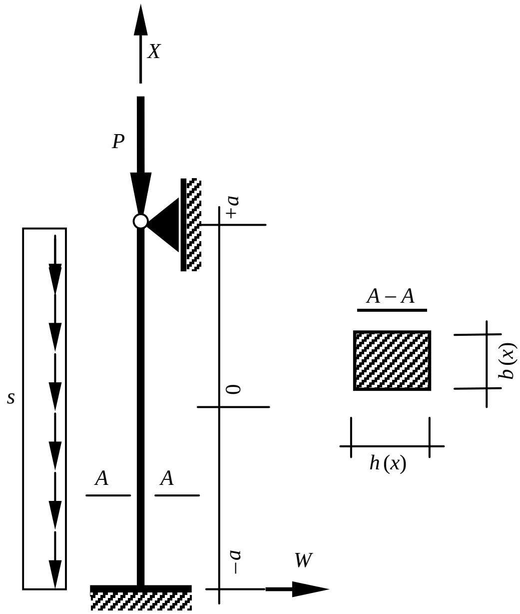

Figure 4. Clamped–simply supported rod loaded with non-potential concentrated force P and uniformly distributed non-potential loads.

4. NUMERICAL EXAMPLES

The above method is now applied to the analysis of the stability of rods under a non-potential load. The examples which are presented below were taken from reference

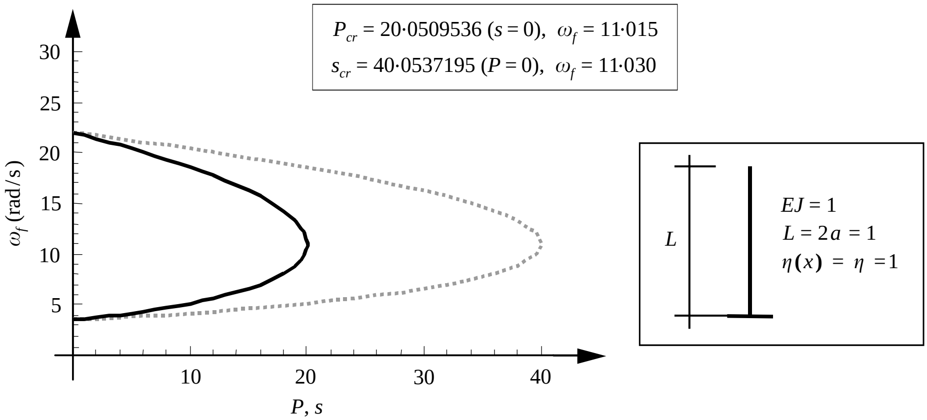

Figure 5. Free vibration frequency versus concentrated follower load P(——) and distributed follower load s (······) for Beck column and Leipholz column.

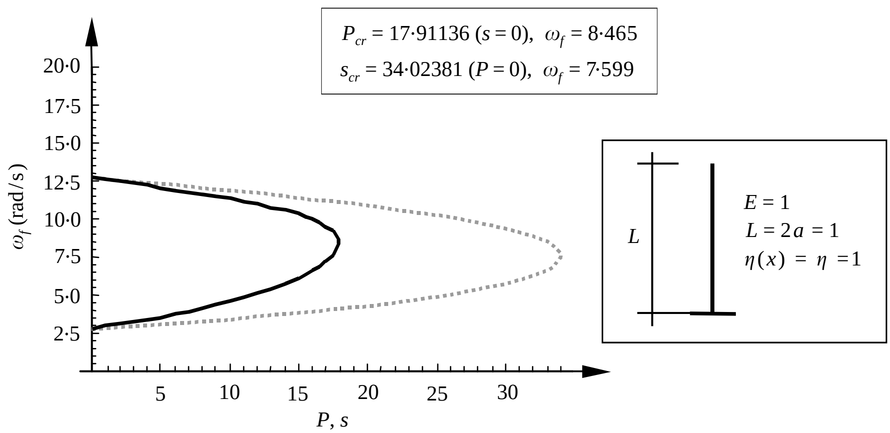

Figure 6. Free vibration frequency versus concentrated follower load P (——) and distributed follower load s (······) for cantilever with variable cross-section $b(x) = h(x) = 2 - (x+1)^2/4$.

[13]. Two static schemes will be considered: a cantilever rod (Figure 3) and a clamped–simply supported rod (Figure 4). The rods are loaded with a concentrated follower force or an evenly distributed tangential follower load. A dynamic stability loss criterion (bifurcation or flatter) is applied to determine the critical values of the considered loads. homogenous rods with variable cross-sections described by functions $b(x)$, $h(x)$ (Figures 3 and 4) are analyzed. It is assumed that: rod length $L = 2a = 1$, Young's modulus $E = 1$ and rod density per unit of volume $\rho_V = 1$. Only one finite element is used to approximate the systems. The boundary conditions needed to solve the problem are as follows:

(1) for the rod shown in Figure 3

$$\phi(-1) = q_1 = 0, \qquad w(-1) = q_3 = 0,$$

(2) for the rod shown in Figure 4

$$\phi(-1) = q_1 = 0, \qquad w(-1) = q_3 = 0, \qquad w(+1) = q_4 = 0.$$

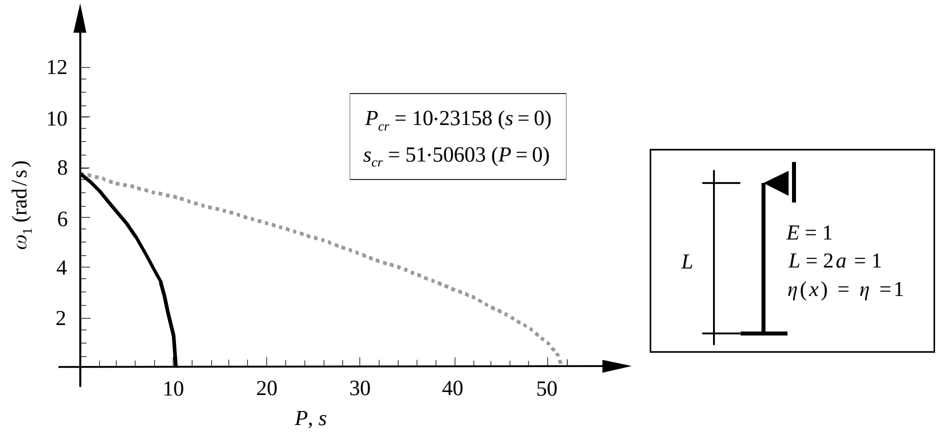

Figure 7. Free vibration frequency versus concentrated follower load P (——) and distributed follower load s (······) for clamped-simply supported rod with variable cross-section $b(x) = h(x) = 2 - (x + 1)^2/4$.

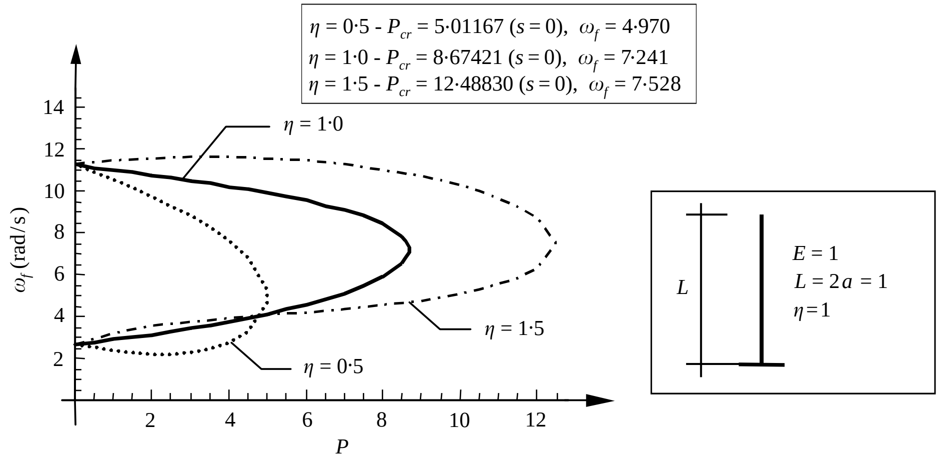

Figure 8. Relationship between critical force and follower coefficient value for Beck column with variable cross-section $b(x) = h(x) = 2 - (x + 1)/2$.

In the first example (Figure 5) a Beck column and a Leipholz column, loaded with a concentrated follower force and an evenly distributed follower load, were analyzed [17, 18]. The solutions of the stability problem for rods with a variable cross-section described by functions $b(x) = h(x) = 2 - (x + 1)^2/4$ are shown in Figures 6 and 7. The influence of follower parameter η on the critical force value for a rod with cross-section $b(x) = h(x) = 2 - (x + 1)/2$ is shown in Figure 8. The determined critical force values and the results obtained by Beck [17], Leipholz and Madan [18] and Glabisz [13] have been compiled in Table 1. The results obtained perfectly agree with the theoretical ones [17, 18] and the ones obtained by approximation in reference [13]. With the use of Chebyshev polynomials, the size of the approximation base can be reduced. In all the examples provided the displacement function was approximated with only 18 Chebyshev series terms. If classic power polynomials are used, this number is much larger (e.g., 70 series elements were used for approximation in reference [13]). To demonstrate as to how the finite elements can be applied to the analysis of more complex beam systems, the problem of

TABLE 1

Comparison of results

	The system shown in Figure 5				The system shown in Figure 6			
	P_{cr}	ω_f	s_{cr}	ω_f	P_{cr}	ω_f	s_{cr}	ω_f
This paper	20·05095	11·015	40·05371	11·030	17·91136	8·465	34·02381	7·599
Reference [13]	20·05095	11·016	40·0537	11·029	17·93	—	34·00	—
References [17, 18]	20·0509 [17]	—	40·05 [18]	—	—	—	—	—
	The system shown in Figure 7				**The system shown in Figure 8**			
	P_{cr}	ω_l	s_{cr}	ω_l	$P_{cr}(\eta = 0{\cdot}5)$	ω_f	$P_{cr}(\eta = 1{\cdot}5)$	ω_f
This paper	10·23158	0·0	51·50603	0·0	5·01167	4·970	12·48830	7·528
Reference [13]	10·23	0·0	51·49	0·0	5·01	—	12·48	—

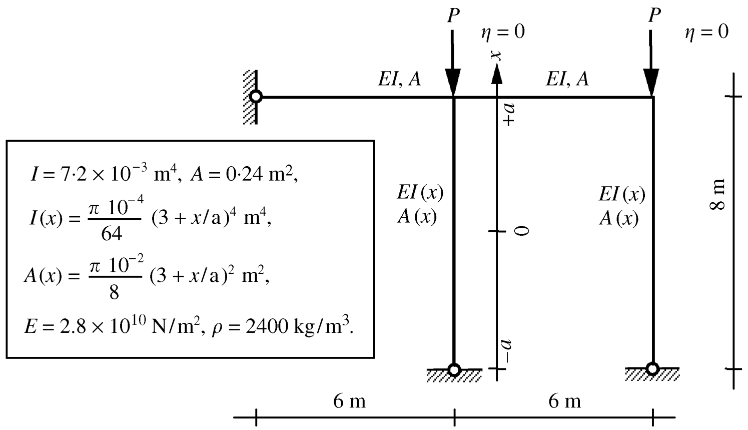

Figure 9. Fixed-joint frame under axial potential load P.

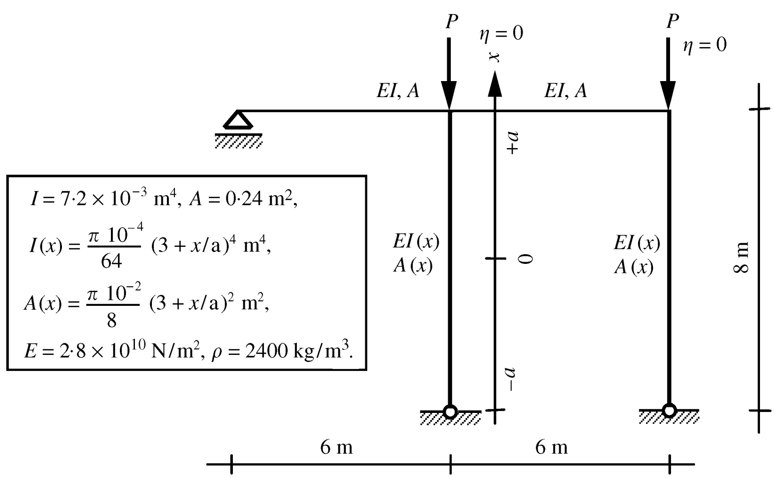

Figure 10. Movable-joint frame under potential load P.

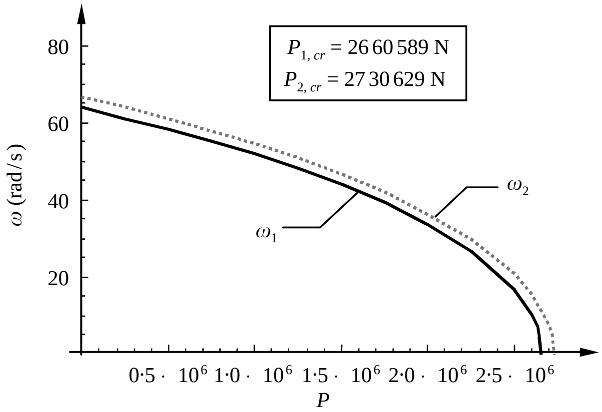

Figure 11. Relationship between frame free vibration (scheme in Figure 9) and axial potential force P.

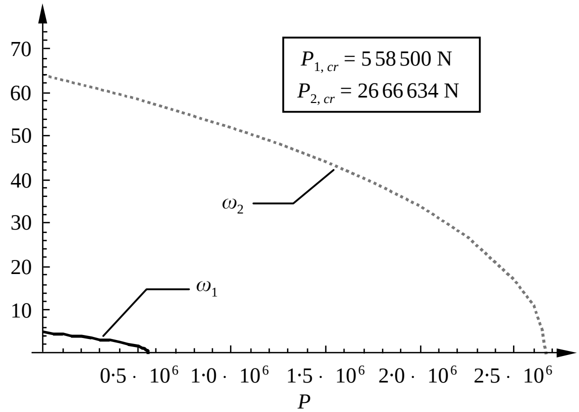

Figure 12. Relationship between frame free vibration (scheme in Figure 10) and axial potential force P.

stability of frame systems under an axial potential load is solved. Schemes of the analyzed frames are shown in Figures 9 and 10. The first example was taken from monograph [1] in which the stability problem was solved by analytical methods for the first critical force only. In the examples shown in Figures 9 and 10, 25 Chebyshev series terms were used to approximate the non-prismatic columns. The results are presented in Figures 11 and 12 and in Tables 2 and 3. For comparison, the results obtained by applying classical finite elements are given in the Table. In the latter case, the system was divided into 12 finite elements (the columns — 4 elements, the spandrel beams — 3 elements). The computations were made using the COSMOS software. As in the previous test tasks, the results were in very good agreement with those obtained by other methods.

5. CONCLUSION

The results obtained prove that the proposed method is effective and therefore useful for determining a dynamic rigidity matrix for non-prismatic rod finite elements. If the exact shape functions yielded by the algorithm presented are applied to the approximation in the

TABLE 2

Comparison of results

	The system shown in Figure 9					The system shown in Figure 10			
	This paper	Classical FEM	This paper	Classical FEM		This paper	Classical FEM	This paper	Classical FEM
P(N)	ω_1(rad/s)	ω_1(rad/s)	ω_2(rad/s)	ω_2(rad/s)	P(N)	ω_1(rad/s)	ω_1(rad/s)	ω_2(rad/s)	ω_2(rad/s)
0	63·96	61·86	66·80	64·38	0	4·77	4·70	64·08	61·98
250 000	61·24	59·10	64·01	61·57	50 000	4·56	4·48	—	61·44
500 000	58·34	56·14	61·05	58·56	100 000	4·33	4·25	—	60·90
750 000	55·21	52·95	57·87	55·32	150 000	4·10	4·01	—	60·34
1 000 000	51·83	49·47	54·43	51·81	200 000	3·84	3·75	—	59·78
1 250 000	48·11	45·63	50·89	47·95	250 000	3·57	3·47	61·36	59·22
1 500 000	43·97	41·32	46·53	43·64	300 000	3·28	3·17	—	58·64
1 750 000	39·26	36·35	41·85	38·72	350 000	2·95	2·82	—	58·06
2 000 000	33·73	30·38	36·41	32·89	400 000	2·57	2·43	—	57·47
2 250 000	26·83	22·60	29·78	25·52	450 000	2·13	1·96	—	56·87
2 500 000	16·94	9·15	20·81	14·35	500 000	1·57	1·31	58·46	56·27

TABLE 3

Comparison of results

	The system shown in Figure 9		The system shown in Figure 10	
	$P_{1,cr}$(N)	$P_{2,cr}$(N)	$P_{1,cr}$(N)	$P_{2,cr}$(N)
This paper	2 660 589	2 730 629	558 500	2 666 634
Classical FEM	2 547 908	2 612 683	540 865	2 553 525
Reference [1]	2 618 000	—	—	—

elements, the size of the problems can be reduced considerably. The numerical examples provided show that even if only one element is used, an exact solution of the stability problem can be obtained. The flatter values and critical bifurcation loads determined by this method are in perfect agreement with those obtained by other authors. Although a simple element shape was used in the examples, the proposed method can be applied to systems made up of rod elements with complex geometry and any distribution of mass and strength parameters, resting on a two-parameter non-homogenous elastic foundation. The simple way of determining the sought coefficients, consisting of the solution of an infinite system of algebraic equations in which the system parameters are described by closed analytical formulas, enables the direct solution of such complex cases.

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